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Extending Leeson's Equation

Matjaž Vidmar

Univerza v Ljubljani, Fakulteta za Elektrotehniko, Ljubljana, Slovenia

Abstract: The oscillator phase noise is one of the key limitations in several fields of electronics. An electronic oscillator phase noise is usually described by the Leeson's equation. Since the latter is frequently misinterpreted and misused, a complete derivation of the Leeson's equation in modern form is given first. Second, effects of flicker noise and active-device bias are accounted for. Next the complete spectrum of an electronic oscillator is derived extending the result of the Leeson's equation into a Lorentzian spectral line. Finally the spectrum of more complex oscillators including delay lines is calculated, like opto-electronic oscillators.

Keywords: phase noise; Leeson's equation; oscillator bias; Lorentzian line; opto-electronic oscillator

Razširitev Leesonove Enačbe

Izvleček: Fazni šum oscilatorja je ena ključnih omejitev v številnih področjih elektronike. Fazni šum elektronskega oscilatorja običajno opisuje Leesonova enačba. Ker je slednja pogosto slabo razumljena in napačno uporabljena, bo najprej opisana celotna izpeljava Leesonove enačbe. V drugem koraku je nujna obravnava učinkov šuma $1/f$ in nastavitve delovne točke aktivnega gradnika. Sledi celovita izpeljava spektra elektronskega oscilatorja, ki rezultat Leesonove enačbe razširi v Lorentzovo spektralno črto. Končno se izpelje spekter bolj kompliciranih oscilatorjev, kot so to opto-elektronski oscilatorji.

Ključne besede: fazni šum; Leesonova enačba; delovna točka oscilatorja; Lorentzova črta; opto-elektronski oscilator

** Corresponding Author's e-mail: matjaz.vidmar@fe.uni-lj.si*

1 Introduction

Towards the end of the 19th century, the Hertz experiments connected two areas of physics, namely electricity and optics. While radio communications started with filtered noise from spark gaps, the latter were quickly replaced by much more efficient vacuum-tube electronic oscillators, invented independently by Armstrong and Meissner around 1912.

Electronic oscillators were so successful that their spectrum was considered an infinitely narrow spectral line at relatively low radio frequencies f<30MHz in the first half of the 20th century. Their spectral line was only broadened by external causes like unfiltered supply, load pull, temperature drift and/or vacuum-tube aging.

On the other hand, in optics it was quickly discovered that spectral lines of different light sources were not infinitely narrow. The optical line width $\Delta\lambda_0$ or Δf could be measured with (relatively simple) interferometers and expressed as longitudinal coherence length d in free space c_0:

$$d \approx \frac{c_0}{\Delta f} \approx \frac{\lambda_0^2}{\Delta\lambda_0} \tag{1}$$

Unfortunately the amplitude dynamic range of simple optical instruments was quite limited.

In the second half of the 20th century, both the frequency resolution of radio measurements as well as the amplitude dynamic range of optical measurements improved by several orders of magnitude. Both keep improving as the user requests keep increasing. Last but not least, the spectrum gap between radio and optics is shrinking as radio frequencies are increasing towards the terahertz region and optical wavelengths are increasing towards the far-infrared region.

One of the most important contributions is the derivation of the oscillator noise spectrum by David Leeson in 1966 [1]. The same derivation is applicable to (relatively low) radio-frequency electronic oscillators as well as to lasers. In electronics, high-performance oscillators are followed by buffer stages that may add their own noise. Electronic limiters may reduce the amplitude noise but they have no effect on the phase noise.

The design of a performing radio-frequency oscillator is complex. Besides basic radio-frequency design the knowledge of different noise contributions is required as well as the knowledge of feedback theory. Due to this complexity the Leeson's equation is frequently misunderstood, misused and even degraded to an "empirical" equation by some sources. The term phase noise only starts appearing in equipment specifications as well as in text books in the 21st century as it is becoming the limiting parameter for increasingly complex modulation schemes at ever increasing carrier frequencies.

2 Electronic oscillator

An electronic oscillator includes an amplifier with a voltage gain A and a feedback network with a voltage transfer function $H(\omega)$. The feedback network is usually a frequency-selective resonator to define the output spectrum of the oscillator:

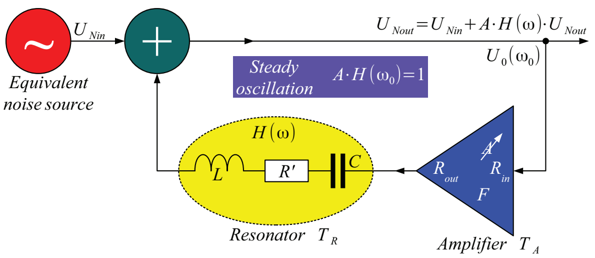

Figure 1: Electronic oscillator.

For the circuit to oscillate, the Barkhausen criterion applies:

$$A\cdot H(\omega_0)=1 \tag{2}$$

The Barkhausen criterion is an equation with complex numbers defining both the phase and the magnitude of the feedback. The circuit can only oscillate at the frequency ω_0 where the feedback phase is zero or an integer multiple of 2π. The amplifier should provide enough gainto start the oscillation. During steady oscillation, saturation will eventually decrease the amplifier gain A to satisfy the Barkhausen criterion.

Some feedback networks may generate complex results. A laser may oscillate at many different modes at the same time. Some electronic circuits may satisfy the Barkhausen criterion at zero frequency. Such circuits do not oscillate but act as bi-stables. A flip-flop intentionally driven into a meta-stable state will quickly settle into one of its two stable states.

Some form of noise is always present in all circuits. In electronic circuits operating in the radio-frequency range, the main contribution is thermal noise. No matter how small, noise will always significantly affect the output spectrum of an oscillator as shown later in the derivation of the Leeson's equation.

In the case of a class A amplifier, noise actually starts the oscillation:

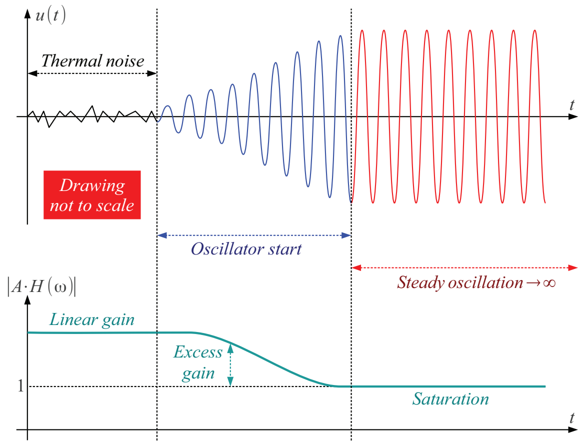

Figure 2: Oscillator start.

With some excess gain, the oscillation amplitude will initially grow exponentially out of noise. As the oscillation amplitude increases, the amplifier will be driven into saturation. The excess gain shrinks and finally reaches the Barkhausen criterion during steady oscillation.

Some oscillators use a class C amplifier. Such oscillators can not start out of noise, but need a start pulse. Unfortunately, after reaching steady oscillation, class C amplifiers add even more noise than class A amplifiers. The gain in class C is lower, there is much less control over the device bias and due to the heavily non-linear operation, class C amplifiers efficiently up-convert low-frequency noise to the desired oscillator frequency.

3 Leeson's equation

The Leeson's equation [1] describes how noise propagates through the circuit of an oscillator. The derivation below refers to Fig 1:

$$U_{Nout}=U_{Nin}+A\cdot H(\omega)\cdot U_{Nout} \tag{3}$$

can be rearranged to:

$$U_{Nout} = \frac{U_{Nin}}{1 - A \cdot H(\omega)} \tag{4}$$

A simple resonator with a lumped capacitor C and a lumped inductor L with losses R' provides the following transfer function of the feedback:

$$H(\omega) = \frac{R_{in}}{R_{in} + j\omega L + R' + \frac{1}{j\omega C} + R_{out}} \tag{5}$$

During steady oscillation the Barkhausen criterion simplifies the transfer function for small signal s $U_{Nout} \ll U_0(\omega_0)$ compared to the carrier to:

$$A \cdot H(\omega) = \frac{\sum R}{\sum R + j\omega L + \frac{1}{j\omega C}} \tag{6}$$

where the sum of resistors denotes:

$$\sum R = R_{in} + R' + R_{out} \tag{7}$$

The transfer function can be further simplified by introducing the loaded quality Q_L of the resonator:

$$Q_L = \frac{\omega_0 L}{\sum R} \tag{8}$$

and the frequency offset from the carrier ω_0:

$$\Delta\omega = \omega - \omega_0 = \omega - \frac{1}{\sqrt{LC}} \tag{9}$$

into:

$$A \cdot H(\omega) \approx \frac{1}{1 + j2Q_L \frac{\Delta\omega}{\omega_0}} \tag{10}$$

resulting in:

$$U_{Nout} \approx \frac{U_{Nin}}{1 - \frac{1}{1 + j2Q_L \frac{\Delta\omega}{\omega_0}}} \rightarrow$$
$$\rightarrow U_{Nout} \approx U_{Nin} \cdot \left(1 + \frac{\omega_0}{j2Q_L \Delta\omega}\right) \tag{11}$$

Dealing with noise is easier with average signal powers $P_j = \alpha\,|U_j|^2$ rather than voltages. The resulting propagation of noise power is:

$$P_{Nout} \approx P_{Nin} \cdot \left[1 + \left(\frac{\omega_0}{2Q_L \Delta\omega}\right)^2\right] \tag{12}$$

In engineering it is also preferred to replace angular frequencies $\omega_j = 2\pi f_j$ with ordinary frequencies:

$$P_{Nout} \approx P_{Nin} \cdot \left[1 + \left(\frac{f_0}{2Q_L \Delta f}\right)^2\right] \tag{13}$$

All derivations in this paper are made considering just one side-band of the symmetrical noise spectrum on both sides of the carrier $U_0(\omega_0)$ or $U_0(f_0)$. If a single side-band is observed, there is no distinction between amplitude noise and phase noise.

When both upper and lower side-bands are summed, the resulting noise signal has both an in-phase component and a quadrature component with respect to the carrier. Due to the random nature of noise, both the in-phase component and the quadrature component are of equal magnitude. The in-phase component adds a random amplitude modulation to the carrier, also called amplitude noise. The quadrature component adds a random phase modulation to the carrier, also called phase noise.

The original Leeson's derivation [1] as well as many other theoretical papers include both noise side-bands, frequently denoted as $S(\omega)$ or $S(f)$. On the other hand, single side-band noise is required in many practical calculations. Care should be taken since both side bands have twice the power of a single side band.

The oscillator noise includes both amplitude noise and phase noise. Both have equal power:

$$P_{NA} = P_{N\phi} = \frac{P_{Nout}}{2} \approx \frac{P_{Nin}}{2} \cdot \left[1 + \left(\frac{f_0}{2Q_L \Delta f}\right)^2\right] \tag{14}$$

Since the amplitude noise P_{NA} can be removed easily with an electronic limiter, only the phase-noise power $P_{N\phi}$ is interesting.

In electronics, noise is usually referred to the input of an amplifier although it can only be measured on its output. Therefore for compatibility all quantities onare referred to the amplifier input. The thermal-noise spectral density dP_{Nin}/df at the amplifier input is equal to the sum of the temperatures of all noise sources multiplied by the Boltzmann constant $k_B \approx 1.38 \cdot 10^{-23}$ J/K:

$$\frac{dP_{Nin}}{df} = k_B \cdot \sum T_j = k_B \cdot \left(T_R + T_A\right) \qquad (15)$$

The resonator temperature $T_R \gg T_0 = 290$ K may be much higher than the reference temperature in the case of resonators using active circuits. The noise temperature of a passive resonator is usually close to the reference (room) temperature $T_R \approx T_0 = 290$ K. In this case the thermal-noise spectral density can be rewritten using the amplifier noise figure F (in linear units!):

$$\frac{dP_{Nin}}{df} \approx k_B \cdot T_0 \cdot F \qquad (16)$$

Note that the amplifier noise figure F will be higher in saturation (steady oscillation) than in linear operation!

The phase-noise spectral density of an oscillator becomes:

$$\frac{dP_{N\phi}}{df} = \frac{1}{2} \cdot \left[1 + \left(\frac{f_0}{2Q_L \Delta f}\right)^2\right] \cdot k_B T_0 F \qquad (17)$$

Since the oscillator output is amplified, limited and/or attenuated, the important quantity is the phase-noise spectral density relative to the oscillator output power P_0:

$$L\left(\Delta f\right) = \frac{1}{P_0} \cdot \frac{dP_{N\phi}}{df} \qquad (18)$$

The relative phase-noise spectral density is denoted by the symbol $L(\Delta f)$ and has units [Hz^{-1}] in the Leeson's equation:

$$L\left(\Delta f\right) = \left[1 + \left(\frac{f_0}{2Q_L \Delta f}\right)^2\right] \cdot \frac{k_B T_0 F}{2P_0} \qquad (19)$$

Due to the extremely wide dynamic range of $L(\Delta f)$ it is common to use logarithmic units, namely decibels relative to the carrier per unit bandwidth or [dBc/Hz]:

$$L\left(\Delta f\right)_{[\mathrm{dBc/Hz}]} = 10\log_{10}\left[L\left(\Delta f\right) \cdot 1\mathrm{Hz}\right] \qquad (20)$$

Unfortunately many popular sources like [2] forget to multiply $L(\Delta f)$ in linear units with the unit bandwidth 1 Hz, degrading the Leeson's equation to an empirical equation.

As an example, the spectrum of a typical oscillator is computed on Fig. 3 using the Leeson's equation. The carrier power is selected as $P_0 = 0.1$ mW typical at the input of a small-signal RF transistor. The noise figure degradation is comparable to the gain compression due to saturation, therefore $F = 10$ dB is a reasonable choice. The most important parameter of an oscillator, the loaded quality of the resonator is selected $Q_L = 10$ corresponding to a varactor-tuned microstrip resonator at $f_0 = 3$ GHz:

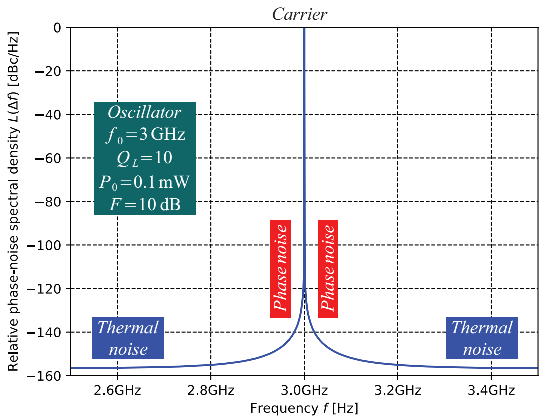

Figure 3: Oscillator spectrum.

The propagation of noise through an oscillator increases the phase noise close to the desired carrier well above the thermal noise. Since the two noise sidebands are symmetric, it makes sense to observe a single side band in detail using a logarithmic scale for the frequency offset Δf from the carrier as shown on Fig. 4:

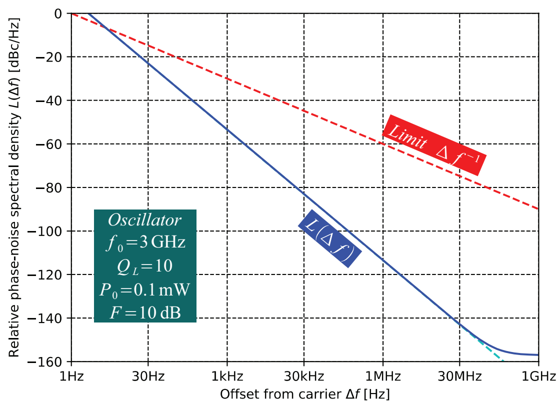

Figure 4: SSB phase-noise spectrum.

At frequency offsets $|\Delta f| > f_0/(2Q_L)$ larger than the Leeson's frequency, the oscillator has little effect on the noise spectral density. Other circuits like buffer amplifiers, limiters and/or attenuators add their own thermal noise. If required, this thermal noise can easily be filtered away using resonators with a similar Q_L as used in the oscillator itself.

At frequency offsets $|\Delta f| < f_0/(2Q_L)$ smaller than the Leeson's frequency, the predominant noise is the oscillator phase noise. Other circuits like amplifiers, limiters and/or attenuators have little effect on the phase-noise spectral density. The oscillator phase noise can NOT be filtered away using resonators with a similar Q_L as used in the oscillator itself.

Since the oscillator phase-noise is the interesting quantity, a simplified Leeson's equation neglecting thermal noise is frequently used:

$$L(\Delta f) \approx \left(\frac{f_0}{Q_L \Delta f}\right)^2 \cdot \frac{k_B T_0 F}{8P_0} \tag{21}$$

The result of the simplified Leeson's equation is shown as a dotted extension on Fig. 4. There is a significant difference from the full equation only at large offsets $|\Delta f| > f_0/(2Q_L) \approx 150\ \mathrm{MHz}$ in the example shown on Fig. 3 and Fig. 4.

The Leeson's equation was derived assuming that the noise amplitude $U_{Nout} \ll U_0\ (\omega_0)$ is much smaller than the desired-carrier amplitude. This assumption no longer holds at small offsets Δf. The Leeson's equation only holds when the relative phase-noise spectral density is much smaller than the $L(\Delta f) \ll \Delta f^{-1}$ limit shown with a dotted line on Fif. 4. In practice, the result on Fig. 4 is only valid at offsets above $|\Delta f| > 1\ \mathrm{kHz}$.

The relative phase-noise density at very small offsets Δf is usually not very important in practical electronic oscillators. It is much more important in laser oscillators. A corrected derivation of the Leeson's equation for very small offsets Δf will be presented later.

4 Effects of phase noise

Phase noise was first noted as residual frequency modulation in analog radio links. The unwanted random frequency deviation (root-mean-square value) can be calculated as:

$$\sigma_f = \sqrt{2\int_{f_{MIN}}^{f_{MAX}} \Delta f^2 L(\Delta f)\, d\Delta f} \tag{22}$$

The frequency limits f_{MIN} and f_{MAX} of the integral are the band limits of the analog base-band modulation signal.

In QAM radio links, phase noise randomly rotates the constellation of the modulation. The unwanted random angle of rotation (root-mean-square value) can be calculated as:

$$\sigma_\phi = \sqrt{2\int_{B_{carrier-recovery}}^{B_{modulation}} L(\Delta f)\, d\Delta f} \tag{23}$$

Any phase noise above $\Delta f > B_{modulation}$ is filtered away by the channel filter in the receiver. Further it is assumed that the carrier-recovery circuit of the receiver is able to track slow frequency and/or phase changes below $\Delta f < B_{carier\text{ - }recovery}$.

In digital communications, phase noise manifests itself as clock jitter. The unwanted clock jitter (root-mean-square value) can be calculated as:

$$\sigma_t = \frac{\sigma_\phi}{\omega_0} = \frac{1}{2\pi f_0}\sqrt{2\int_{B_{clock-recovery}}^{f_{MAX}} L(\Delta f)\, d\Delta f} \tag{24}$$

Limiting the bandwidth of the clock, the upper limit $f_{MAX} < f_0$ is less than the clock frequency. Further it is assumed that the clock-recovery circuit of the receiver is able to track slow frequency and/or phase changes below $\Delta f < B_{clock\text{ - }recovery}$.

Finally in all radio communications, phase noise causes interference to neighbor channels. The interference power can be calculated as:

$$P_i = P_0 \cdot \int_{\Delta f_1}^{\Delta f_2} L(\Delta f)\, d\Delta f \tag{25}$$

The frequency limits Δf_1 and Δf_2 of the integral are the frequency offsets of the interfered channel from the interfering carrier $P_0(f_0)$.

Note that all of the above-mentioned integrals start from an offset $\Delta f > 0$ larger than zero. Radio equipment is usually designed to work with relatively clean sources where the phase-noise power $P_{N\phi} \ll P_0$ is much smaller than the carrier power and the Leeson's equation is valid thanks to $L(\Delta f) \ll \Delta f^{-1}$ in the region of interest.

5 Active-device noise

Besides thermal noise, active devices also add flicker noise to the amplified signal. Flicker noise is usually described as an increase of the radio-frequency noise figure F into a frequency-dependent noise figure $F'(f)$:

$$F'(f) = F \cdot \left(1 + \frac{f_C}{f}\right) \tag{26}$$

The parameter describing flicker noise is the corner frequency f_C. The latter depends on the device technology [3]. In general, surface devices have higher current densities and more structure defects than bulk devices. Surface semiconductor devices like a silicon MOSFET, a GaAs MESFET or a GaAlAs HEMT may have the corner frequency in the range $f_C \approx 1 \ldots 10$ MHz. Bulk semiconductor devices like a silicon BJT or a silicon JFET may have the corner frequency in the range $f_C \approx 1 \ldots 10$ kHz.

Although a HEMT may produce slightly less noise at radio frequencies than a BJT, a HEMT is significantly noisier at low frequencies than a BJT as shown on Fig. 5:

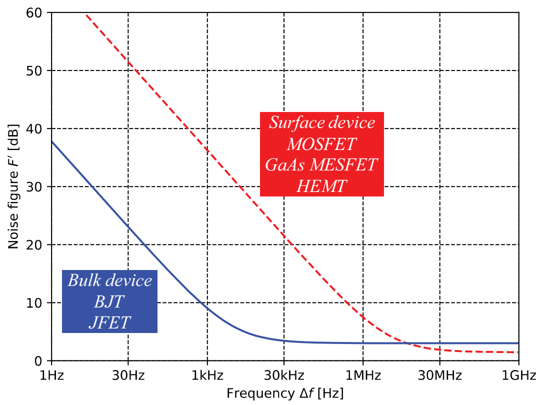

Figure 5: Active device noise figure.

In an oscillator, the active device operates in saturation while producing steady oscillations. The nonlinear effects associated with saturation up-convert the low-frequency flicker noise into noise side bands very close to the carrier radio frequency. High-performance radio-frequency (microwave) oscillators therefore use silicon bipolar transistors due to their lower flicker noise.

The additional up-converted flicker noise can be built into the Leeson's equation describing the increase the oscillator phase noise at small offsets $|\Delta f| < f_C$:

$$L(\Delta f) = \left[1 + \left(\frac{f_0}{2Q_L \Delta f}\right)^2\right] \cdot \frac{k_B T_0 F}{2P_0} \cdot \left(1 + \frac{f_C}{|\Delta f|}\right) \tag{27}$$

The phase noise of the same oscillator example as shown earlier including flicker noise is shown on Fig 6:

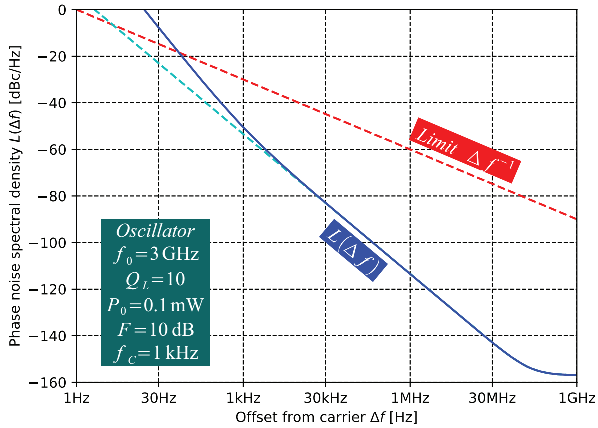

Figure 6: Phase noise including flicker noise.

Calculations including flicker noise may not be simple. Calculating the flicker-noise power P_N from equation (26):

$$P_N = \int_{f_{MIN}}^{f_{MAX}} k_B \cdot F \cdot \left(1 + \frac{f_C}{f}\right) df \tag{28}$$

may give an infinite result:

$$\lim_{f_{MIN} \to 0} \int_{f_{MIN}}^{f_{MAX}} k_B \cdot F \cdot \left(1 + \frac{f_C}{f}\right) df \to \infty \tag{29}$$

suggesting that further limitations apply to (26) at very low frequencies.

Further it is necessary to understand that the flicker-noise corner frequency f_C in equation (26) is different from the f_C in equation (27)! Between the two quantities there is a frequency conversion that may be more or less efficient depending on parameters that are NOT described by the Leeson's equation!

The phase noise of an oscillator depends heavily on the bias and DC decoupling circuits. Since the impedance parameters $[Z_{ij}]$ of a bipolar transistor depend mainly on the DC currents through the device, the currents through the RF amplifier transistor have to be regulated as constant as possible with a bias circuit like that on Fig. 7 [4]. Keeping the impedance parameters $[Z_{ij}]$ constant attenuates the up-conversion of low-frequency flicker noise to the RF carrier frequency:

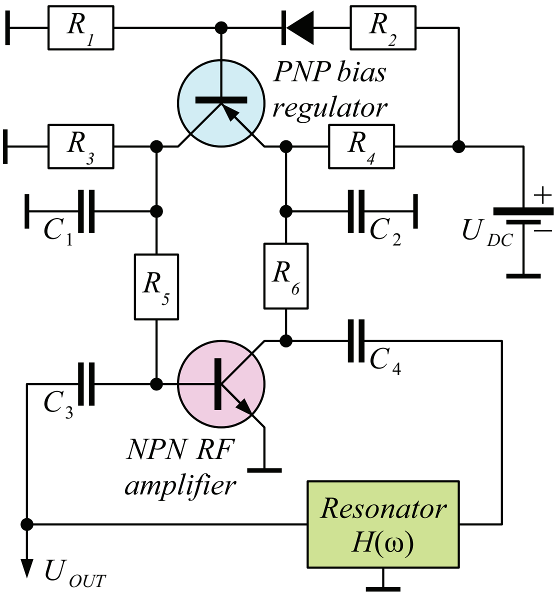

Figure 7: Oscillator bias circuit.

Flicker noise is not the only concern while designing the bias network of an oscillator. Reactive components like RF chokes (inductors) may introduce additional unwanted modes of the resonator $H(\omega)$. Therefore resistors R_5 and R_6 are usually used to apply the DC bias in oscillators.

Besides the RF feedback there is yet another feedback circuit built into every electronic oscillator. Gain reduction at saturation during steady oscillation is governed by this additional feedback (bottom graph on Fig. 2). A poorly-designed bias network will make this low-frequency feedback unstable causing self quenching of the oscillator. While self quenching may simplify a super-regenerative receiver compared to the original Armstrong design [9], it has a catastrophic effect on the oscillator spectrum.

The gain-reduction feedback already has one pole due to the RF energy stored in the resonator $H(\omega)$, rectified by the nonlinear effects of the saturation of the active device and added to the DC bias of the latter. Additional poles are added by the RF bypass capacitors C_1 and C_2 and by the DC-bias decoupling capacitors C_3 and C_4. Unless the component values on Fig. 7 are selected carefully, the oscillator will be self-quenching. Even if the oscillator is not self-quenching, a poor phase margin of the bias feedback may cause a significant increase of the oscillator phase noise.

If varactors are used to tune the oscillator (VCO) [6], the phase noise is degraded further. First, varactors decrease the Q_L of the resonator due to their series resistance. Second, the tuning voltage may introduce additional noise. Even the noise voltage introduced by the resistors acting as RF chokes to tune the varactors is not insignificant.

6 Spectral-line width

The Leeson's equation (19) is unable to describe the frequency spectrum of an oscillator very close to its central frequency ω_0 or f_0 when the condition $L(\Delta f) \ll \Delta f^{-1}$ is no longer fulfilled. Although there are several comprehensive papers on this topic like [5], [6], a simplified derivation is given here.

Analyzing Fig. 1, the feedback gain has to be slightly less than unity during steady oscillation, since some noise is being added all of the time. Accordingly, the original Barkhausen criterion (2) has to be modified to:

$$A \cdot H(\omega_0) = 1 - \epsilon \tag{30}$$

where the gain decrease is described by the very small, but non-zero quantity $0<\epsilon\ll 1$. The feedback transfer function (10) is modified to:

$$A \cdot H(\omega) = \frac{1-\epsilon}{1 + j2Q_L \dfrac{\Delta\omega}{\omega_0}} \tag{31}$$

resulting in equation (11) extended to:

$$U_{Nout} = \frac{U_{Nin}}{1 - A \cdot H(\omega)} \approx \frac{U_{Nin}}{1 - \dfrac{1-\epsilon}{1 + j2Q_L \dfrac{\Delta\omega}{\omega_0}}} \rightarrow$$
$$\rightarrow U_{Nout} \approx U_{Nin} \cdot \frac{1 + j2Q_L \dfrac{\Delta\omega}{\omega_0}}{j2Q_L \dfrac{\Delta\omega}{\omega_0} - \epsilon} \tag{32}$$

At frequency offsets $|\Delta f| > f_0/(2Q_L)$ larger than the Leeson's frequency, the oscillator has little effect on the noise while other circuits add their own noise. It therefore makes sense to evaluate (32) at small offsets $|\Delta f| < f_0/(2Q_L)$ only. Considering $|j2Q_L \Delta\omega_0/\omega_0| \ll 1$, equation (32) simplifies to:

$$U_{Nout} \approx \frac{U_{Nin}}{j2Q_L \dfrac{\Delta\omega}{\omega_0} - \epsilon} \tag{33}$$

Replacing noise voltages with average powers, replacing angular frequencies with ordinary frequencies and considering the phase noise only:

$$P_{N\phi} = \frac{P_{Nout}}{2} \approx \frac{P_{Nin}/2}{\left(2Q_L \frac{\Delta f}{f_0}\right)^2 + \epsilon^2} \quad (34)$$

Introducing the thermal-noise spectral density (15) or (16) and the spectral-line half width:

$$f_{HW} = \frac{\epsilon f_0}{2Q_L} \quad (35)$$

the simplified Leeson's equation (21) evolves into a Lorentzian spectral line:

$$L(\Delta f) = \left(\frac{f_0}{Q_L}\right)^2 \cdot \frac{1}{\Delta f^2 + f_{HW}^2} \cdot \frac{k_B T_0 F}{8P_0} \quad (36)$$

The missing quantities f_{HW} or ϵ can be calculated by summing the whole relative spectrum power considering $\Delta f = f - f_0$:

$$\int_{-f_0}^{\infty} L(\Delta f)\, d\Delta f = 1 \quad (37)$$

In all practical cases the integral start may be replaced by – ∞, the error being smaller than neglecting far-away thermal noise:

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\frac{f_0}{Q_L}\right)^2 \cdot \frac{1}{\Delta f^2 + f_{HW}^2} \cdot \frac{k_B T_0 F}{8P_0}\, d\Delta f = \\ &= \left(\frac{f_0}{Q_L}\right)^2 \cdot \frac{k_B T_0 F}{8P_0} \left[\frac{1}{f_{HW}} \cdot \arctan \frac{\Delta f}{f_{HW}}\right]_{\Delta f=-\infty}^{\Delta f=\infty} = \\ &= \left(\frac{f_0}{Q_L}\right)^2 \cdot \frac{k_B T_0 F}{8P_0} \cdot \frac{\pi}{f_{HW}} \approx 1 \end{aligned} \quad (38)$$

The spectral-line half width is obtained as:

$$f_{HW} \approx \pi \cdot \left(\frac{f_0}{Q_L}\right)^2 \cdot \frac{k_B T_0 F}{8P_0} \quad (39)$$

The small correction of the Barkhausen criterion is:

$$\epsilon \approx \frac{\pi f_0 k_B T_0 F}{4 Q_L P_0} \quad (40)$$

Analyzing the same oscillator example with $f_0 = 3$ GHz, $Q_L = 10$, $P_0 = 0.1$ mW and $F = 10$ dB as on Fig. 3 and Fig. 4, a spectral-line half width of $f_{HW} \approx 14\text{Hz}$ is obtained. The corresponding correction of the Barkhausen criterion is small indeed $\epsilon \approx 10^{-7}$.

One side band of the calculated spectrum $L(\Delta f)$ (solid line) is compared to the original Leeson's equation (dotted extensions) on Fig. 8 in logarithmic scale:

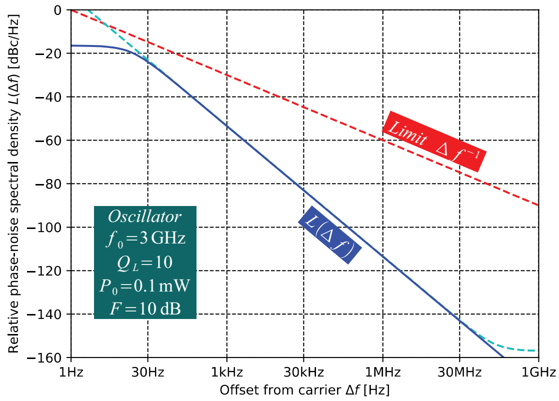

Figure 8: Lorentzian spectral line.

The result of the original Leeson's equation is plotted with a dotted line on the same graph as well as the Δf^{-1} limit. Note that at small offsets the spectrum $L(\Delta f)$ flattens thus avoiding the Δf^{-1} limit.

Besides thermal noise, additional noise like flicker noise further broadens the spectral line. The calculation is more difficult since the low-frequency flicker-noise spectrum is not up-converted by a single carrier frequency but by the oscillator signal itself with non-zero spectral width.

In most cases the spectral-line half width remains much narrower $f_{HW} \ll B_{recovery}$ than the carrier or clock recovery circuits in radio equipment. In all these frequent cases the result of the original Leeson's equation is sufficient.

7 Delay-line oscillators

The most important parameter in the Leeson's equation is the loaded quality Q_L of the resonator. Unfortunately electrical resonators in the radio-frequency range do not achieve very high values of Q_L. Mechanical resonators like quartz crystals are frequently used in high-performance radio oscillators. Electrical resonators may achieve very high values of Q_L in the optical-frequency range. Lasers may produce relatively very narrow spectral lines. Unfortunately dividing optical frequencies down to radio frequencies is not practical yet.

Delay lines may act as resonators in oscillator circuits. Their equivalent Q_{LD} is directly proportional to the delay τ_D and increases linearly with frequency:

$$Q_{LD} = \pi f_0 \tau_D \tag{41}$$

Unfortunately delay lines may fulfill the Barkhausen criterion (2) at many different frequencies causing a laser to oscillate on many different modes. Lasers may use frequency-selective mirrors or gain medium to decrease the number of modes.

A similar approach may be used to design radio-frequency oscillators using either acoustic (BAW or SAW) delay lines or opto-electronic delay lines [7]. The latter look promising due to the low loss and wide bandwidth of optical fibers. The basic design of an opto-electronic oscillator is shown on Fig. 9. The desired mode of oscillation is selected by an additional electric (microwave) resonator:

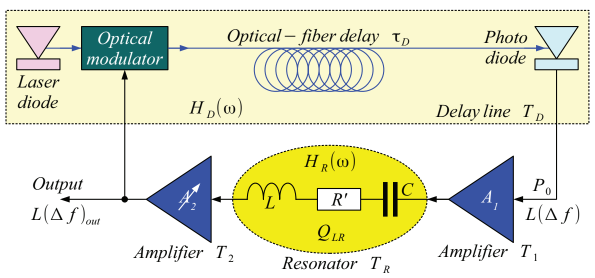

Figure 9: Opto-electronic oscillator.

The Barkhausen criterion (2) can be rewritten for the circuit on Fig. 9 as:

$$A_1 \cdot H_R(\omega_0) \cdot A_2 \cdot H_D(\omega_0) = 1 \tag{42}$$

If the electric resonator is tuned precisely to the desired mode of the delay line, the voltage transfer function of the latter can be written as:

$$H_D(\omega) = a \cdot \mathrm{e}^{-j\Delta\omega\tau_D} \tag{43}$$

For small signals and small offsets:

$$A_1 \cdot H_R(\omega) \cdot A_2 \cdot H_D(\omega) = \frac{\mathrm{e}^{-j\Delta\omega\tau_D}}{1 + j2Q_{LR}\dfrac{\Delta\omega}{\omega_0}} \tag{44}$$

The noise-voltage transfer function becomes:

$$U_{Nout} \approx \frac{U_{Nin}}{1 - \dfrac{\mathrm{e}^{-j\Delta\omega\tau_D}}{1 + j2Q_{LR}\dfrac{\Delta\omega}{\omega_0}}} \tag{45}$$

The corresponding phase-noise average power is:

$$P_{N\phi} \approx \frac{P_{Nin}}{2} \cdot \left| 1 - \frac{\mathrm{e}^{-j\Delta\omega\tau_D}}{1 + j2Q_{LR}\dfrac{\Delta\omega}{\omega_0}} \right|^{-2} \tag{46}$$

Finally the extended Leeson's equation for the opto-electronic oscillator shown on Fig. 9 becomes:

$$L(\Delta f) = \frac{k_B \sum T_j}{2P_0} \cdot \left| 1 - \frac{\mathrm{e}^{-j2\pi\Delta f\tau_D}}{1 + j2Q_{LR}\dfrac{\Delta f}{f_0}} \right|^{-2} \tag{47}$$

The largest contribution to ΣT_j comes from the opto-electronic delay line that may include flicker noise:

$$\sum T_j \approx T_D \cdot \left(1 + \frac{f_C}{|\Delta f|} \right) \tag{48}$$

In an opto-electronic oscillator as on Fig. 9 the most vulnerable point in the circuit is the photo-diode output. Here the signal power P_0 is the lowest and the relative phase-noise spectral density $L(\Delta f)$ is calculated. Saturation will likely be achieved in A_2 since optical modulators require substantial amounts of RF drive power. The output $L(\Delta f)_{out}$ is taken after all amplification and filtering:

$$L(\Delta f)_{out} \approx \frac{L(\Delta f)}{1 + \left(2Q_{LR}\dfrac{\Delta f}{f_0} \right)^2} \tag{49}$$

The analytical result for $L(\Delta f)_{out}$ is fitted to the well-documented experimental data from [8]. The latter describes a microwave $f_0 = 3$ GHz opto-electronic oscillator with the delay line made from $l \approx 15$km of optical fiber resulting in a delay of $\tau_D \approx 75\mu$s corresponding to a $Q_{LD} \approx 7 \cdot 10^5$. Mode selection is performed by an additional microwave dielectric resonator with the $Q_{LR} \approx 8300$.

The opto-electronic delay line noise temperature may be rather high due to several reasons: relative intensity noise (RIN) of the laser, optical reflections including Rayleigh scattering converting optical phase noise into amplitude noise and inefficient broadband impedance matching of the photodiode. The opto-electronic delay line noise temperature is found as expected around $T_D \approx 2 \cdot 10^5$ K. What really matters is the ratio T_D/P_0 and the latter can be measured conveniently at the output of a PIN-FET module.

Flicker noise comes at least in part from the built-in HEMT amplifiers. Due to the required relatively high electronic gain, several amplifier stages are connected in series. If broadband amplifiers are used, flicker noise may originate in the first stage, it is amplified by the intermediate stage and it is up-converted by the last stage. Due to the high noise contribution from the opto-electronic delay line, the overall flicker-noise corner frequency is found around $f_C \approx 5$ kHz.

The fitted analytical result for $L(\Delta f)_{out}$ on Fig. 10 shows the unwanted side modes at the correct frequencies. However, the peak magnitudes of the unwanted modes are about 15 dB stronger than the measured values. This may be due to an insufficient resolution of the phase-noise test setup:

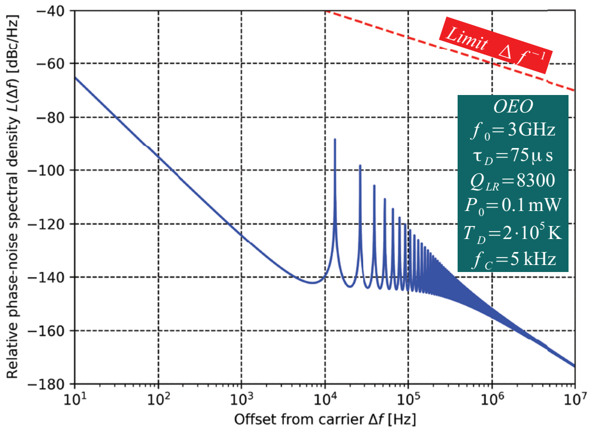

Figure 10: Simulated OEO phase noise.

The well-documented experimental data from [8] additionally includes results with a Q-multiplier circuit. The latter increases the loaded quality of the microwave mode-selection filter to about $Q_{LR} \approx 75000$ thus improving the rejection of unwanted modes. Since a Q multiplier is an active filter, the system noise temperature increases to about $T_D \approx 5 \cdot 10^5$ K.

The fitted analytical result for $L(\Delta f)_{out}$ including the Q multiplier is shown on Fig. 11. The unwanted-mode magnitudes are reduced and their line widths are broader. Both frequencies and magnitudes are very close to the measured values in [8]:

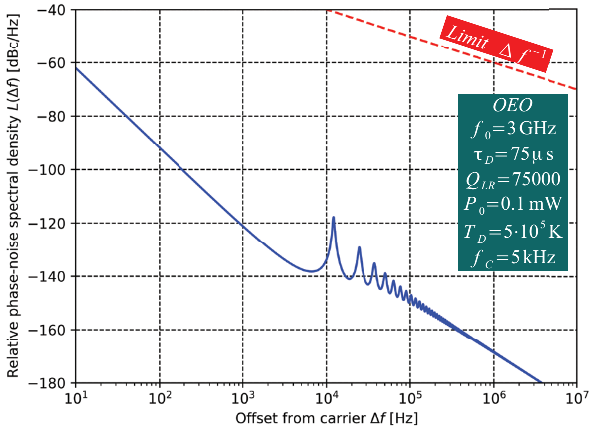

Figure 11: Simulated OEO with Q multiplier.

Finally, a parabolic approximation of the close-in response of a single microwave resonator suggests that the unwanted mode rejection is proportional to $(Q_{LR})^4$. For a Q-multiplication factor $m \approx 8$ as described in [8], the unwanted-mode rejection improvement is expected as $10\log_{10} m^4 \approx 36$ dB. The difference between Fig. 10 and Fig. 11, corrected for the change in T_D, comes much closer to this value than the measured data published in [8], again suggesting an insufficient resolution of the phase-noise test setup.

8 Avoiding UV & DC catastrophes

When natural laws are extended from a few laboratory measurements up to the whole frequency spectrum, problems usually arise at both extremes: when the frequency approaches infinity $f \to \infty$ and when the frequency approaches zero $f \to 0$. One of the most famous problems in physics was the ultraviolet catastrophe predicted from the Rayleigh-Jeans law for black-body thermal radiation [10], suggesting infinite radiated power. The more accurate Planck's law solved the problem a few years later.

The same problem also applies to phase noise. What happens with the relative phase noise density at both extremes $L(\Delta f \to \infty)$ (UV catastrophe) and $L(\Delta f \to 0)$ (DC catastrophe)? The answer is not simple since $L(\Delta f)$ may achieve very differing shapes and magnitudes. To the best of my knowledge, the limitations of different equations for phase noise that may produce non-physical results are identified by introducing the Δf^{-1} limit for the first time in this article.

The Leeson's equation for electronic oscillators is usually derived from the Johnson noise. In electronics, the Johnson-noise spectral density is usually considered frequency-independent as shown in equation (15), since it is derived form the Rayleigh-Jeans law. At room temperatures, the Rayleigh-Jeans law becomes inaccurate at infrared frequencies. At cryogenic temperatures, the Rayleigh-Jeans law becomes inaccurate already at microwave-radio frequencies.

If equation (15) is rewritten to include the complete Planck's law, the resulting thermal-noise spectral density also depends on the Planck constant $h \approx 6.626 \cdot 10^{-34}$ Js:

$$\frac{dP_N}{df}\left[\frac{\mathrm{W}}{\mathrm{Hz}}\right] = \frac{hf}{e^{\frac{hf}{k_B T}} - 1} \tag{50}$$

The Johnson noise is just an approximation for low frequencies:

$$\frac{dP_N}{df}\left(hf \ll k_B T\right) \approx \lim_{f \to 0}\left(\frac{hf}{e^{\frac{hf}{k_B T}} - 1}\right) = k_B T \tag{51}$$

The complete equation should be considered at frequencies above $f \approx k_B T/h \approx 6$ THz at a room temperature $T \approx 290$ K. The electronic noise decays even sooner since the gain bandwidth of electronic devices is about three orders of magnitude smaller. Therefore there are at least two valid and independent reasons to avoid the UV catastrophe.

Flicker noise is usually modeled as $1/f$ noise in equation (26). The latter suggests an infinite amount of power (DC catastrophe) even in a simple amplifier without feedback (29). If the spectral noise density at very low frequencies or the total noise power is required, a better model than $1/f$ should be used for flicker noise. In order to separate different effects, flicker noise will not be considered in the following discussion.

Another DC catastrophe may originate in the simple derivation of the Leeson's equation for an oscillator (19) or (21). At small offsets $\Delta f \rightarrow 0$, the noise power is no longer small compared to the carrier power. A complete derivation of the spectral line (36) avoids this DC catastrophe.

In order to explain different effects, the same result from Fig. 8 is plotted on much broader scales on Fig. 12. On the latter, the frequency spans from bi-weekly 10^{-6} Hz up to soft X rays 10^{18} Hz. The amplitude range spans an incredible 350 dB:

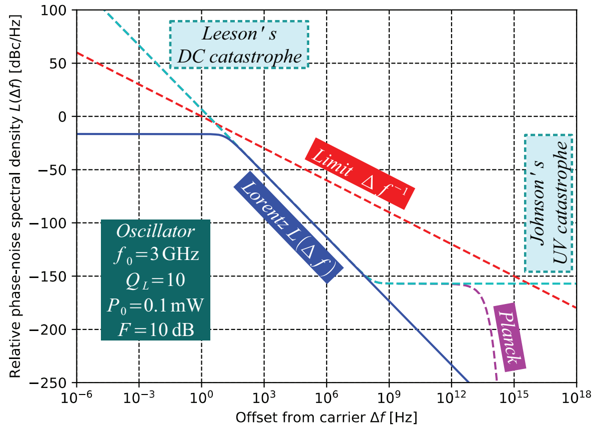

Figure 12: Catastrophes explained.

The Δf^{-1} limit corresponds to an infinite amount of power over the whole spectrum. Considering both side-bands of a single octave $f < \Delta f < 2f$, the Δf^{-1} limit produces a finite amount, just slightly too much relative noise power:

$$\frac{P_N}{P_0} = 2\int_f^{2f} \Delta f^{-1} d\Delta f = 2\ln 2 \approx 1.386 \tag{52}$$

In order to comply with equation (37), the relative phase-noise spectral density $L(\Delta f)$ may approach the Δf^{-1} limit over less than an octave and drop to zero elsewhere else.

A Lorentzian spectral line approaches the Δf^{-1} limit to -8 dB at a frequency offset $\Delta f = f_{HW}$(≈ 14 Hz) (39). At smaller offsets $\Delta f \ll f_{HW}$ the Lorentzian spectral line is flat with frequency $L(\Delta f) \approx \alpha$. At larger offsets $\Delta f \gg f_{HW}$ the Lorentzian spectral line decays as $L(\Delta f) \approx \alpha \cdot \Delta f^{-2}$ with increasing offset. At both smaller and larger offsets Δf, the Lorentzian spectral line diverges far below the Δf^{-1} limit thus avoiding both UV and DC catastrophes.

The result of the Leeson's equation (dotted line) matches the Lorentzian spectral line (solid line) over the usually-interesting offset range and stays well below the Δf^{-1} limit. At very small offsets $\Delta f \rightarrow 0$, the Leeson's result grows as $L(\Delta f) \approx \alpha \cdot \Delta f^{-2}$ with decreasing offset, eventually exceeding the Δf^{-1} limit and causing a DC catastrophe. At very large offsets $\Delta f \rightarrow \infty$, the Leeson's result is flat with frequency $L(\Delta f) \approx \alpha$, eventually exceeding the Δf^{-1} limit and causing an UV catastrophe.

As long as the relative phase-noise spectral density $L(\Delta f)$ is a monotonically decreasing function, it should avoid the Δf^{-1} limit in a similar way to the Lorentzian spectral line. More complex spectra $L(\Delta f)$ like that shown on Fig. 10 may even exceed the Δf^{-1} limit over very narrow offset ranges (much less than an octave) without causing catastrophes analyzing a likely useless oscillator.

In any case, comparing the magnitude and slope of $L(\Delta f)$ to the Δf^{-1} limit quickly tells whether a certain equation for $L(\Delta f)$ with certain parameters provides useful results or not over the desired offset range. The ratio $L(\Delta f)/\Delta f^{-1} = \Delta f \cdot L(\Delta f)$ tells whether the phase noise at the specified offset Δf is much smaller or comparable to the whole signal power.

9 Conclusions

The Leeson's equation for relative phase-noise spectral density is frequently misunderstood and misused even in commercial simulation software. Therefore a complete derivation is made first to understand the limitations of the different forms of the same equation. While derivations produce results in linear units $[\mathrm{Hz}^{-1}]$, logarithmic units $[\mathrm{dBc/Hz}]$ (20) are used elsewhere including the graphs in this article.

The complete Leeson's equation (19) is frequently simplified to (21), since wide-band thermal noise originates elsewhere and not just in the oscillator.

Flicker noise is usually built in the Leeson's equation like (27), but its exact magnitude actually depends on factors not included in the Leeson's equation, like the design of active-device bias networks. Last but not least, the simple $1/f$ approximation of flicker noise may produce non-physical, infinite results in some cases.

The original Leeson's derivation is valid for small noise signals only. The result is only valid in the offset range when $L(\Delta f) \ll \Delta f^{-1}$. When $L(\Delta f)$ approaches or even exceeds the Δf^{-1} limit, non-physical results are usually obtained. In the latter case a complete derivation of the oscillator spectrum has to be performed including the shape of the main spectral line of non-zero width. Flat thermal noise produces a Lorentzian spectrum (36).

Finally, the Leeson's equation is extended to delay-line oscillators and in particular to opto-electronic oscillators. The extended equation (47) is fitted to experimental data showing potential problems of the latter.

As a conclusion of all of the above findings, an electronic oscillator is just a Q multiplier amplifying and filtering its own noise. The Q-multiplication factor is very large $m \approx \epsilon^{-1}$ resulting in a very small, but non-zero spectral-line half width $f_{HW} > 0$. Besides bandwidth differences of many orders of magnitude, an electronic oscillator produces a similar signal to the spark radio transmitter or filtered white light in optics.

10 Conflict of Interest

The author declares no conflict of interest.

The founding sponsors had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, and in the decision to publish the results.

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